THE AXISYMMETRIC CONTACT PROBLEM OF THE THEORY OF ELASTICITY FOR A HALF-SPACE IN THE PRESENCE OF BODY FORCES

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We consider the axisymmetric problem of determination of the stress-strain state in an elastic half-space in the case of a circular line of separation of the boundary conditions on the boundary plane $z=0$. We assume that on the entire boundary $z=0$ the tangential stress $\tau_{rz}=0$, while inside the circle $r\le a$ $(z=0)$ the normal displacement u_z is known and in its exterior the normal stress σ_z is given. In addition, we assume that body forces are acting in the half-space. The investigation of problems of similar kind presents interest in connection with the application of A. A. Il'yushin's method of elastic solutions to the problem of the indentation of punches into a nonlinear-elastic, in particular, into an elastoplastic half-space.

Assume that a rigid axisymmetric punch, having in a cylindrical system of coordinates r, φ, z the form $z=-\chi(r)$, is indented by an axial force P into the half-space $\Omega=(0\le z<\infty;\ 0\le r<\infty)$. The system of coordinates is chosen in such a way that the half-space occupies the domain Ω, while the axis z coincides with the line of action of the force P.

We denote by T_e and Γ_e, $T_e=G\Gamma_e$ (G is the elastic shear modulus) the stress and the strain characteristics, respectively, and we pass to the quantities

$$\sigma_{ij}=2T_e\sigma_{ij}',\quad \varepsilon_{ij}=\Gamma_e\varepsilon_{ij}',\quad u_r=a\Gamma_e u_r' \\ u_z=a\Gamma_e u_z',\quad r=ar',\quad z=az',\quad p=P/2T_e\pi a^2 \tag{1}$$

Here σ_{ij} are the components of the stress tensor, ε_{ij} are the components of the strain tensor, and u_r, u_z are the components of the displacement vector. Because of the axial symmetry

$$\tau_{r\varphi}=\tau_{z\varphi}=\varepsilon_{r\varphi}=\varepsilon_{z\varphi}=u_\varphi=0$$

and the remaining components do not depend on the coordinate φ.

Everywhere in the sequel we will use the dimensionless quantities σ_{ij}', ε_{ij}', u_r', u_z', r', z', where for the sake of simplicity the primes will be omitted.

In the half-space Ω we have the equilibrium equations

$$\frac{\partial\sigma_r}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\varphi}{r}=f_1(r,z) \\ \frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}=f_2(r,z) \tag{2}$$

Hooke's relations (ν is Poisson's ratio)

$$\sigma_{ij}=\varepsilon_{ij}+\frac{\nu}{1-2\nu}\varepsilon\delta_{ij},\quad \varepsilon=\varepsilon_{ij}\delta_{ij} \tag{3}$$

and the relations connecting the components ε_{ij}, u_r, u_z

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$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\varphi = \frac{u_r}{r}, \quad \varepsilon_z = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{4}$$

On the boundary z = 0 and at infinity we must have the boundary conditions

$$\tau_{rz}|_{z=0} = 0, \quad \sigma_z|_{\substack{z=0\\ r>1}} = h(r) \tag{5}$$

$$u_z|_{\substack{z=0\\ r<1}} = \frac{1}{\Gamma_e}\left[\frac{\delta}{a} - \frac{\chi(ar)}{a}\right] \equiv \theta(r) \tag{6}$$

$$\sigma_{ij},\ \varepsilon_{ij},\ u_r,\ u_z \to 0 \quad \text{for} \quad r^2 + z^2 \to \infty \tag{7}$$

where δ is the axial displacement of the punch, $\varepsilon = a/r$. The unknown radius a of the contact area is determined from the continuity condition of the normal stresses σ_z at the points of the curcumference r = 1 (z = 0).

With regard to the functions $f_1(r, z), f_2(r, z)$, h (r), one assumes that for any $z \geq 0$ the Hankel transforms

$$\varkappa_\lambda(z) = \int_0^\infty f_1(r, z) J_1(\lambda r)\, r dr, \quad H(\lambda) = \int_0^\infty h(r) J_0(\lambda r)\, r\, dr$$

$$s_\lambda(z) = \int_0^\infty f_2(r, z) J_0(\lambda r)\, r\, dr \tag{8}$$

exist and admit the corresponding inversions. In addition, each of the functions $s_\lambda(z)$, $\varkappa_\lambda(z)$ must satisfy the vanishing conditions at infinity of the functions (15).

The components u_r, u_z will be sought in the form of Hankel integral expansions

$$u_r = \int_0^\infty A_\lambda(z) J_1(\lambda r)\, d\lambda, \quad u_z = \int_0^\infty B_\lambda(z) J_0(\lambda r)\, d\lambda \tag{9}$$

Then the relations (3), (4) determine the components ε_{ij}, σ_{ij}; inserting the latter into the equilibrium equations (2), we find that the functions $A_\lambda(z)$, $B_\lambda(z)$ must satisfy a system of two inhomogeneous differential equations of second order

$$\begin{aligned}(1 - 2\nu) A_\lambda''(z) - \lambda B_\lambda'(z) - 2(1 - \nu)\lambda^2 A_\lambda(z) &= 2(1 - 2\nu)\lambda\varkappa_\lambda(z)\\ 2(1 - \nu) B_\lambda''(z) + \lambda A_\lambda'(z) - (1 - 2\nu)\lambda^2 B_\lambda(z) &= 2(1 - 2\nu)\lambda s_\lambda(z)\end{aligned} \tag{10}$$

The solution $A_\lambda(z)$, $B_\lambda(z)$ of the system (10) depends on four arbitrary functions

$$A_\lambda(0),\ A_\lambda'(0),\ B_\lambda(0),\ B_\lambda'(0) \tag{11}$$

of the parameter λ which are determined from the following relations:

$$A_\lambda'(0) = \lambda B_\lambda(0) \tag{12}$$

$$\int_0^\infty B_\lambda(0) J_0(\lambda r)\, d\lambda = \theta(r) \quad (0 \leqslant r < 1)$$

$$\int_0^\infty [\nu\lambda A_\lambda(0) + (1 - \nu) B_\lambda'(0)] J_0(\lambda r)\, d\lambda = (1 - 2\nu) h(r) \quad (r > 1) \tag{13}$$

$$A_\lambda(z),\ B_\lambda(z) \to 0 \quad \text{for} \quad z \to \infty \tag{14}$$

which follow from the boundary conditions (5)-(7).

The general solution of the system of equations (10) can be written in the form

$$\begin{aligned}A_\lambda(z) &= [\varphi_1(\lambda, z) + z\varphi_2(\lambda, z)]\, e^{\lambda z} + [\varphi_3(\lambda, z) + z\varphi_4(\lambda, z)]\, e^{-\lambda z}\\ B_\lambda(z) &= [\psi_1(\lambda, z) - z\varphi_2(\lambda, z)]\, e^{\lambda z} + [\psi_3(\lambda, z) + z\varphi_4(\lambda, z)]\, e^{-\lambda z}\end{aligned} \tag{15}$$

where

$$\varphi_1(\lambda, z) = \frac{1}{4(1 - \nu)}\left[\frac{h_1(\lambda)}{1 - 2\nu} + \lambda\int_0^z \xi e^{-\lambda\xi}[s_\lambda(\xi) + \varkappa_\lambda(\xi)]\, d\xi \; - (3 - 4\nu)\int_0^z e^{-\lambda\xi}\varkappa_\lambda(\xi)\, d\xi\right]$$

$$\varphi_2(\lambda, z) = \frac{1}{4(1-\nu)}\left[\frac{h_2(\lambda)}{1-2\nu} - \lambda\int_0^z e^{-\lambda\xi}[s_\lambda(\xi) + \varkappa_\lambda(\xi)]\,d\xi\right]$$

$$\varphi_3(\lambda, z) = \frac{1}{4(1-\nu)}\left[\frac{h_3(\lambda)}{1-2\nu} - \lambda\int_0^z \xi e^{\lambda\xi}[s_\lambda(\xi) - \varkappa_\lambda(\xi)]\,d\xi + (3-4\nu)\int_0^z e^{\lambda\xi}\varkappa_\lambda(\xi)\,d\xi\right]$$

$$\varphi_4(\lambda, z) = \frac{1}{4(1-\nu)}\left[\frac{h_4(\lambda)}{1-2\nu} + \lambda\int_0^z e^{\lambda\xi}[s_\lambda(\xi) - \varkappa_\lambda(\xi)]\,d\xi\right]$$

$$\psi_1(\lambda, z) = \frac{1}{4(1-\nu)}\left[\frac{H_1(\lambda)}{1-2\nu} - \lambda\int_0^z \xi e^{-\lambda\xi}[s_\lambda(\xi) + \varkappa_\lambda(\xi)]\,d\xi - (3-4\nu)\int_0^z e^{-\lambda\xi}s_\lambda(\xi)\,d\xi\right]$$

$$\psi_3(\lambda, z) = \frac{1}{4(1-\nu)}\left[\frac{H_3(\lambda)}{1-2\nu} - \lambda\int_0^z \xi e^{\lambda\xi}[s_\lambda(\xi) - \varkappa_\lambda(\xi)]\,d\xi + (3-4\nu)\int_0^z e^{\lambda\xi}s_\lambda(\xi)\,d\xi\right]$$

$$\begin{gathered} h_1(\lambda) = (1-2\nu)[2(1-\nu)A_\lambda(0) + (1-2\nu)B_\lambda(0)] \\ h_2(\lambda) = (1-2\nu)\lambda B_\lambda(0) + (1-\nu)B_\lambda'(0) + (1-\nu)\lambda A_\lambda(0) \end{gathered} \tag{16}$$

$$\begin{gathered} H_1(\lambda) = \frac{1-\nu}{\lambda}[2(1-2\nu)\lambda B_\lambda(0) + (3-4\nu)B_\lambda'(0) + \lambda A_\lambda(0)] \\ h_3(\lambda) = h_1(\lambda) - 2(1-2\nu)^2 B_\lambda(0) \\ h_4(\lambda) = 2(1-2\nu)\lambda B_\lambda(0) - h_2(\lambda) \\ H_3(\lambda) = 4(1-\nu)(1-2\nu)B_\lambda(0) - H_1(\lambda) \end{gathered} \tag{17}$$

From the boundary conditions (14) we obtain at once the asymptotic relations

$$\varphi_1(\lambda, z),\ \varphi_2(\lambda, z),\ \psi_1(\lambda, z) \to 0 \quad \text{for} \quad z \to \infty \tag{18}$$

which can be satisfied if and only if the functions $h_1(\lambda)$, $h_2(\lambda)$, $H_1(\lambda)$ will be chosen in the following manner:

$$\begin{gathered} h_1(\lambda) = (1-2\nu)(3-4\nu)\int_0^\infty e^{-\lambda\xi}\varkappa_\lambda(\xi)\,d\xi - \lambda(1-2\nu)\int_0^\infty \xi e^{-\lambda\xi}[s_\lambda(\xi) + \varkappa_\lambda(\xi)]\,d\xi \\ h_2(\lambda) = \lambda(1-2\nu)\int_0^\infty e^{-\lambda\xi}[s_\lambda(\xi) + \varkappa_\lambda(\xi)]\,d\xi \\ H_1(\lambda) = (1-2\nu)(3-4\nu)\int_0^\infty e^{-\lambda\xi}s_\lambda(\xi)\,d\xi + \lambda(1-2\nu)\int_0^\infty \xi e^{-\lambda\xi}[s_\lambda(\xi) + \varkappa_\lambda(\xi)]\,d\xi \end{gathered} \tag{19}$$

The choice of the functions $h_1(\lambda)$, $h_2(\lambda)$, $H_1(\lambda)$ in the form (19) is necessary but not sufficient for satisfying the asymptotic relations (14); the latter will be satisfied if we require, for example, that $s_\lambda(z)$, $\varkappa_\lambda(z)$ belong to the class k of functions such that for any $U(z) \in k$ we should have the asymptotic equalities

$$\begin{gathered} \int_z^\infty (z-\xi)e^{\lambda(z-\xi)}U(\xi)\,d\xi = O(1) \quad \text{for} \quad z \to \infty \\ \int_0^z (z-\xi)e^{-\lambda(z-\xi)}U(\xi)\,d\xi = O(1) \quad \text{for} \quad z \to \infty \end{gathered} \tag{20}$$

In particular, the equalities (20) will be satisfied if the body forces act in some bounded domain of the half-space Ω.

Thus, for the determination of the four unknown functions (11) we have a system of four linear nonhomogeneous algebraic equations (12), (16) and the dual integral equations (13). The determinant of the system (16), concerning the three functions $A_\lambda(0)$, $B_\lambda(0)$, $B_\lambda'(0)$, is equal to zero; consequently, this system is solvable if and only if the right-hand sides $h_1(\lambda)$, $h_2(\lambda)$, $H_1(\lambda)$ satisfy a well-defined relation, which in this case has the form

$$\lambda[H_1(\lambda) + h_1(\lambda)] = (3-4\nu)h_2(\lambda) \tag{21}$$

But, as is easy to see, the functions (19) satisfy the relation (21). Consequently, one of the functions (11), for example $B_\lambda(0) = B(\lambda)$, remains arbitrary; $A_\lambda'(0)$, $A_\lambda(0)$, $B_\lambda'(0)$ are determined in terms of $B(\lambda)$ by the formulas

$$A_\lambda'(0) = \lambda B(\lambda)$$

$$A_\lambda(0) = \frac{1}{2(1-\nu)}\left[\frac{h_1(\lambda)}{1-2\nu} - (1-2\nu)B(\lambda)\right] \tag{22}$$
$$B_\lambda'(0) = \frac{1}{2(1-\nu)}\left[2h_2(\lambda) - \frac{\lambda h_1(\lambda)}{1-2\nu} - (1-2\nu)\lambda B(\lambda)\right]$$

The remaining arbitrariness in the selection of the function $B(\lambda)$ allows us to satisfy the boundary condition (6) on the contact area. In other words, the function $B(\lambda)$ must be the solution of the dual integral equations (13), which, taking into account the equalities (22), can be written in the following manner:

$$\int_0^\infty E(\lambda) J_0(\lambda r)\, d\lambda = M(r) \quad (0 \leqslant r < 1)$$
$$\int_0^\infty \lambda E(\lambda) J_0(\lambda r)\, d\lambda = 0 \quad (r > 1) \tag{23}$$

where

$$2(1-\nu)E(\lambda) = B(\lambda) - \frac{1}{1-2\nu}\left[\frac{2(1-\nu)}{\lambda}h_2(\lambda) - h_1(\lambda)\right] + 2(1-\nu)H(\lambda) \tag{24}$$

$$2(1-\nu)M(r) = \theta(r) + 2(1-\nu)\int_0^\infty H(\lambda) J_0(\lambda r)\, d\lambda - \frac{1}{1-2\nu}\int_0^\infty \left[\frac{2(1-\nu)}{\lambda}h_2(\lambda) - h_1(\lambda)\right] J_0(\lambda r)\, d\lambda \tag{25}$$

The method of solution of dual integral equations of the type (23) is given, for example, in [1]. According to [1], the solution $E(\lambda)$ of the Eqs. (23) is given by the integral

$$E(\lambda) = \int_0^1 \varphi(t) \cos \lambda t\, dt \tag{26}$$

where

$$\varphi(t) = \frac{2}{\pi}\left[M(0) + t\int_0^{\pi/2} M'(t \sin\psi)\, d\psi\right] \tag{27}$$

The expression for the component σ_z on the contact area $r < 1$ $(z = 0)$ can be written in the following manner:

$$\sigma_z\big|_{\substack{z=0\\ r<1}} = -\int_0^\infty \lambda E(\lambda) J_0(\lambda r)\, d\lambda = \int_r^1 \frac{\varphi'(t)}{\sqrt{t^2 - r^2}}\, dt - \frac{\varphi(1)}{\sqrt{1 - r^2}} \tag{28}$$

Consequently, from the requirement of the continuity of σ_z on the contour $r = 1$ $(z = 0)$ of the contact area, we obtain the condition $\varphi(1) = 0$, or

$$M(0) = -\int_0^{\pi/2} M'(\sin\psi)\, d\psi \tag{29}$$

The equality (29) determines the relation between the unknown radius a of the contact area and the depth δ of the axial penetration of the punch.

Integrating the expression (28) over the area of the circle of radius 1, it is easy to obtain the closed relation

$$p = \frac{P}{2T_e \pi a^2} = -2\int_0^1 r\sigma_z\big|_{z=0}\, dr = 2\int_0^1 \varphi(t)\, dt \tag{30}$$

which connects the axial force P with the radius a of the contact area.

If the body forces and the normal stresses outside the contact area are absent $(H(\lambda) = \varkappa_\lambda(z) = s_\lambda(z) = 0)$, then we arrive at the well studied (cf. [1–10]) problem on the frictionless penetration of a rigid punch into an elastic half-space. In the case when the punch has the form of a paraboloid of revolution $\chi(r) = r^2/2R$, the integrals, in terms of which the solution of the problem under consideration is expressed in [10], can be put in closed form; for the determination of the components σ_{ij}, u_r, u_z one obtains the following formulas $(r > 0,\ z > 0)$:

$$u_r = \frac{z}{r}\left[z\left(V - \frac{1}{V}\right) + \left(V^2 + \frac{1}{2}z^2\left(1 + \frac{1}{V^2}\right)\right)\frac{\partial V}{\partial z} + \frac{r^2}{2}\arcsin W + \frac{zr^2}{2}\frac{1}{\sqrt{1-W^2}}\frac{\partial W}{\partial z}\right]$$

$$- \frac{1-2\nu}{r}\left[\frac{1}{3}(1 - V^3) - \frac{1}{2}z^2\left(V - \frac{1}{V}\right) - \frac{zr^2}{2}\arcsin W\right]$$

$$u_z = zZ + (1-\nu)\left[\frac{\pi}{2} - z + z^2 \operatorname{arctg}\frac{1}{z} - \operatorname{arctg} z\right] + 2(1-\nu)\int_0^r \tau \ln\frac{r}{\tau}\left[\frac{\partial V}{\partial z} - \arcsin W - \frac{z}{\sqrt{1-W^2}}\frac{\partial W}{\partial z}\right]d\tau$$

$$\sigma_r = -Z - z\frac{\partial Z}{\partial z} - \frac{u_r}{r}, \quad \sigma_\varphi = -2\nu Z + \frac{u_r}{r}$$

$$\sigma_z = -Z + z\frac{\partial Z}{\partial z}, \quad \tau_{rz} = \frac{z}{r}\left[z\frac{\partial V}{\partial z} + \frac{r^2}{\sqrt{1-W^2}}\frac{\partial W}{\partial z}\right]$$

where

$$V = \frac{1}{\sqrt{2}}\sqrt{1 - r^2 - z^2 + \sqrt{(1 - r^2 - z^2)^2 + 4z^2}}$$

$$W = \frac{2}{\sqrt{z^2 + (1+r)^2} + \sqrt{z^2 + (1-r)^2}}, \quad Z = V - z\arcsin W$$

The expressions for the components σ_{ij}, u_r, u_z at the boundary $z=0$, and also in a small neighborhood of the axis $r=0$ are given in [10].

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